Exact unsteady solution to the n-dimensional compressible Navier-Stokes equations

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Abstract

Exact explicit analytical vortex and potential solutions to the n-dimensional inviscid Burgers equations in the Euler variables for an inertial movement of fluid particles in unbounded space is obtained. An exact analytical solution to the one-dimensional compressible Euler equations in the explicit form instead of the Riemann implicit solution is stated. The corresponding example of an exact solution for the main turbulence theory problem on its basis is resolved. An example for exact analytical solution with finite integral of energy for 3-D unsteady compressible Navier-Stokes equations in the unbounded space that is regular over arbitrary large time only when any finite viscous is taken into account is obtained. Onsager’s dissipative anomaly problem, which is also known as Kolmogorov’s hypothesis about an independence of the turbulence energy dissipation rate on the Reynolds number in the inertial sub-range of scales only in the six-dimensional space may be resolved exactly.

Basic publications:

1. S. G. Chefranov, An exact statistical closed description of vortex turbulence and the diffusion in compressible medium, **Sov. Phys. Dokl. 36(4), 286 (1991)**
2. S. G. Chefranov, A. S. Chefranov, Exact solution of the compressible Euler-Helmholtz equation and the millennium prize problem generalization, Phys. Scr., 94, 054001 (2019); https://doi.org/10.1088/1402-4896/aaf918
3. S. G. Chefranov, A. S. Chefranov, The new exact solution of the compressible 3D Navier-Stokes equations, Commun. Nonlinear Sci. Numer. Simul., 83, 205118 (2020); https://doi.org/10.1016/j.cnsns.2019.105118
4. S. G. Chefranov, A. S. Chefranov, Exact solution to the main turbulence problem for a compressible medium and the universal -8/3 law turbulence spectrum of breaking waves, Phys. Fluids, 33, 076108 (2021); https://doi.org/10.1063/5.005621
5. S. G. Chefranov, A. S. Chefranov, Universal turbulence scaling law -8/3 at fusion implosion, Phys. Fluids 34, 036105 (2022); https://doi.org/10.1063/5.0082164
6. A. S. Chefranov, S. G. Chefranov, G. S. Golitsyn, Cosmic rays self-arising turbulence with universal spectrum -8/3, Astrophysical J., 951:38 (2023); https://doi.org/10.3847/1538-4357/acd53a

The founders of modern turbulence theory

FIG. 4. Lev Landau in the late 1920s (first; photo credit: Ryndina, 2004); Andrey Nikolaevich Kolmogorov on board the *Dimitry Mendelyeev* (second; photo credit: Shiryaev, 1999, frontispiece) and Alexander Mikhailovich Oboukhov (third; photo credit: Yaglom, 1990); Evgeny Alekseevich Novikov (fourth; photo credit: Evgeny Alekseevich Novikov); and Robert Harry Kraichnan (fifth; photo credit: Chen *et al.*, 2008).

John Z. Shi
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The formulation to the main problem of turbulence theory-T-problem according to A. Monin and A. Yaglom:

Establishing a mathematically closed description of the time evolution of any moments for the hydrodynamic fields satisfying the equations of hydrodynamics of a compressible medium is the unresolved up to now big T-problem.

E. A. Novikov, 1976 : Method of randomization for an exact solutions in hydrodynamics when the characteristic Hopf functional, that gives the solution of T-problem, is obtained directly.

But how to obtain exact unsteady solution to the 3-D Helmholtz vortex equation?

E. A. Novikov, 1978: It is need to obtain explicit analytical solution of 3-D Hopf's equations that gives the class of solutions to the 3-D Helmholtz vortex equations.

The 3-D Hopf's and 3-D Helmholtz's vortex equations

n-D Riemann-Hopf $$\frac{\partial u_i}{\partial t} + u_l \frac{\partial u_i}{\partial x_l} = 0; i = 1, 2, .., n \quad (1)$$

What is the explicit form for solution of Eq. (1) when its implicit form is well-known?

$$\boldsymbol{Implicit\ solution}: u_i(\vec{x}; t) = u_{i0}\left(\vec{x} - t\vec{u}(\vec{x}; t)\right)$$

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \equiv rot\vec{u}$$

3-D Helmholtz $$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} - \omega_i divu \quad (2)$$

Exact Hopf's functional solution (E. A. Novikov, S. G. Chefranov, 1978; S. G. Chefranov, PhD 1979)

In particular, for the 2-D stationary periodic two-dimensional flow of incompressible medium, the corresponding solution for the characteristic Hopf functional is obtained:

$$\boldsymbol{u_i(\vec{x}) = \varepsilon_{ij}\frac{\partial\psi}{\partial x_j}\,;\,\psi = -ak^{-1}\cos\left(\vec{k}\vec{x}+\theta\right)};$$

$$\boldsymbol{U_{ij}(\vec{k}) = \frac{1}{(2\pi)^2}\int d^2r\left\langle u_i(\vec{x}+\vec{r})u_j(\vec{x})\right\rangle exp\left(-i\vec{k}\vec{r}\right) = \frac{\langle a^2\rangle}{2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)Q(\vec{k})};$$

$$\boldsymbol{\Phi[z] \equiv \left\langle exp\left\{i\int d^2xz(\vec{x})\psi(\vec{x})\right\}\right\rangle =}$$

$$\boldsymbol{\int da\,S(a)\int d^2kQ(\vec{k})J_0\left(\frac{a}{k}\left(\tilde{z}(\vec{k})\tilde{z}(-\vec{k})\right)^{1/2}\right)} \qquad (3)$$

Euler's and the Riemann-Hopf equations

The Euler equations for a compressible medium:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}; i = 1, .., n \qquad (4)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0; \qquad (5)$$

$$\int d^n x \left[\frac{1}{2}\left(\frac{\partial u^2}{\partial t} + u_i \frac{\partial u^2}{\partial x_i}\right) - c^2\left(\frac{\partial \ln \rho}{\partial t} + div u\right)\right] = 0 \qquad (6)$$

In the limit of large Mach numbers, $M = |\vec{u}|/c >> \sqrt{2}$ (here, $c^2 = \partial p/\partial \rho$), the Euler equation (1), with an accuracy of up to discarded terms of the order of $O\left(\frac{1}{M^2}\right)$:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = 0.$$

Exact explicit analytical solutions of the n-D Hopf equation (1)

Implicit solution of (1): $\boldsymbol{u_i(\vec{x};t) = u_{0i}(\vec{x} - t\vec{u}(\vec{x};t))}$

Its Fourier's transformation:

$$\boldsymbol{\tilde{u}_i(\vec{k};t) = \int d^n x u_i(\vec{x};t)\, exp\left(i\vec{k}\vec{x}\right) = \int d^n x\, u_{0i}(\vec{x} - t\vec{u}(\vec{x};t))\, exp(\, i\vec{k}\vec{x})} \quad (7)$$

Let us introduce the new variables: $\boldsymbol{\vec{\xi} = \vec{x} - t\vec{u}(\vec{x};t) \equiv \vec{x} - t\vec{u}_0(\vec{\xi})}$ (8)

Explicit representation for Fourier's transformation of the n-D Hopf equation solution:

$$\boldsymbol{\tilde{u}_i(\vec{k};t) = \int d^n \xi\, u_{0i}(\vec{\xi})\, \widehat{det\, A}\,(\vec{\xi};t)\, exp\left(i\vec{k}\left(\vec{\xi} + t\vec{u}_0(\xi)\right)\right)} \quad (9)$$

Only for the case of **1-D (n=1)**, when $\widehat{\boldsymbol{det\,A}} = \mathbf{1} + \boldsymbol{t}\frac{\boldsymbol{du_{01}(\xi_1)}}{\boldsymbol{d\xi_1}}$ the explicit solution in the form (9) is known (V. Kuznetsov 1969; Rudenko, 1974; E. Pelinovsky 1976 etc.)

$$\boldsymbol{\delta(\vec{x}) = \frac{1}{(2\pi)^n}\int d^n k\, exp(-i\vec{k}\vec{x})} \qquad (10)$$

$$\boldsymbol{u_i(\vec{x};t) = \frac{1}{(2\pi)^n}\int d^n k\, \widetilde{u}_i\left(\vec{k};t\right) exp\left(-i\vec{k}\vec{x}\right) =}$$

$$\boldsymbol{\int d^n \xi u_{0i}\left(\vec{\xi}\right) \widehat{det\,A}\, \delta\left(\vec{\xi} - \vec{x} + t\vec{u}_0\left(\vec{\xi}\right)\right)};$$

$$\boldsymbol{\widehat{det\,A} \equiv det\,A_{ij} > 0; A_{ij} = \delta_{ij} + t\frac{\partial u_{0i}(\vec{\xi})}{\partial \xi_j}} \quad \mathbf{(11)}$$

Explicit analytical solution for 3-D Helmholtz vortex equations (Chefranov 1991):

Let us use the following relations, which are satisfied by the delta function:

$$\frac{\partial \delta(y)}{\partial x_m} \equiv -A_{km}^{-1}(\xi, t)\frac{\partial \delta(y)}{\partial \xi_k}; y_i = \xi_i - x_i + t u_{0i}(\xi); A_{li}A_{im}^{-1} = \delta_{lm} \qquad (12)$$

$$\delta(\xi_1 - \xi + t(u_0(\xi_1) - u_0(\xi))) \equiv \frac{\delta(\xi_1 - \xi)}{det\, A(\xi, t)} \qquad (13)$$

$$\frac{\partial u_{0m}}{\partial \xi_k} A_{km}^{-1}\, det\, A = \frac{\partial\, det\, A}{\partial t} \qquad (14)$$

$$\frac{\partial}{\partial \xi_k}\left(A_{km}^{-1}\, det\, A\right) \equiv 0 \qquad (15)$$

An explicit form to the exact solutions of the n-D continuity equation (5) and of the 3-D vorticity equation (2) are obtained:

$$\boldsymbol{\rho(x,t) = \int d^n\xi \rho_0(\xi)\delta(\xi - x + tu_0(\xi))\,; i = 1,2,3,\ldots,n} \quad (16)$$

$$\boldsymbol{\omega_i(x,t) = \int d^3\xi \left(\omega_{0i}(\xi) + t\omega_{0j}\frac{\partial u_{0i}}{\partial \xi_j}\right)\delta(\xi - x + tu_0(\xi)); i = 1,2,3 \omega = rotu; \omega_0 = rotu_0; E_0 = \frac{1}{2}\int d^3x u_0^2 < \infty} \quad (17)$$

Solutions (11) and (16), (17) are conserved there smoothness only over a finite time interval $\boldsymbol{0 \leq t \leq t_C}$:

$$\boldsymbol{\widehat{det\,A} \equiv det\left(\delta_{ij} + t\frac{\partial u_{0i}(\vec{x})}{\partial x_j}\right) > 0} \quad (18)$$

The finite time of the solution breaking

The minimum lifetime of a smooth solution is determined by finding the solution of the equation

$$\widehat{\boldsymbol{det\,A}}\,(\boldsymbol{t};\vec{\boldsymbol{u}}_{\boldsymbol{0}}(\vec{\boldsymbol{x}})) = \boldsymbol{0}\ :\boldsymbol{n} = \boldsymbol{1}\!: \widehat{\boldsymbol{det\,A}} = \boldsymbol{1} + \boldsymbol{t}\,\frac{\boldsymbol{du_{01}(x_1)}}{\boldsymbol{dx_1}} = \boldsymbol{0};$$

$$\boldsymbol{t_C} = \frac{\boldsymbol{1}}{\boldsymbol{max_{x_1}|du_{01}/dx_1|}} \qquad (19)$$

$$\boldsymbol{n} = \boldsymbol{2}\!: \widehat{\boldsymbol{det\,A}} = \boldsymbol{1} + \boldsymbol{t\,div}\vec{\boldsymbol{u}}_{\boldsymbol{0}} + \boldsymbol{t^2}\left(\frac{\boldsymbol{\partial u_{01}}}{\boldsymbol{\partial x_1}}\frac{\boldsymbol{\partial u_{02}}}{\boldsymbol{\partial x_2}} - \frac{\boldsymbol{\partial u_{01}}}{\boldsymbol{\partial x_2}}\frac{\boldsymbol{\partial u_{02}}}{\boldsymbol{\partial x_1}}\right) = \boldsymbol{0};$$

$$\boldsymbol{n} = \boldsymbol{3}\!: \widehat{\boldsymbol{det\,A}} = \boldsymbol{1} + \boldsymbol{t\,div}\vec{\boldsymbol{u}}_{\boldsymbol{0}} + \boldsymbol{t^2 A_1} + \boldsymbol{t^3 A_2} = \boldsymbol{0},$$

$$\boldsymbol{A_1} = \boldsymbol{U_{012}} + \boldsymbol{U_{013}} + \boldsymbol{U_{023}}, \boldsymbol{U_{012}} \equiv \frac{\boldsymbol{\partial u_{01}}}{\boldsymbol{\partial x_1}}\frac{\boldsymbol{\partial u_{02}}}{\boldsymbol{\partial x_2}} - \frac{\boldsymbol{\partial u_{01}}}{\boldsymbol{\partial x_2}}\frac{\boldsymbol{\partial u_{02}}}{\boldsymbol{\partial x_1}};\ \boldsymbol{A_2} = \boldsymbol{det}\left(\frac{\boldsymbol{\partial u_{0i}}}{\boldsymbol{\partial x_j}}\right)$$

For the 1-D: $\boldsymbol{u_{01} = a\,exp\left(-\frac{x_1^2}{2L^2}\right)}$

$$\boldsymbol{t_C = \frac{L\sqrt{e}}{a}; x_1 = x_{1M} = L} \qquad \textbf{(20)}$$

For the 2-D: $\boldsymbol{div\vec{u}_0 = 0; u_{0i} = \varepsilon_{ij}\frac{\partial\psi_0}{\partial x_j}; \psi_0 = a\sqrt{L_1L_2}\,exp\left(-\frac{x_1^2}{L_1^2} - \frac{x_2^2}{L_2^2}\right)}$:

$$\boldsymbol{t_C = \frac{1}{max_{x_1;x_2}\sqrt{U_{012}}} = \frac{e\sqrt{L_1L_2}}{2a}; \frac{x_{1M}^2}{L_1^2} + \frac{x_{2M}^2}{L_2^2} = 1} \qquad \textbf{(21)}$$

Exact solution to the 3-D compressible Navier-Stokes equations (N-S-E), regular at any time

Let us consider the modification of n-D Riemann-Hopf equations which gives N-S-E modeling in the form:

$$\frac{\partial u_i}{\partial t} + \left(u_j(\vec{x};t) + V_j(t)\right)\frac{\partial u_i}{\partial x_j} = -\mu u_i;\ \mu = \frac{\nu}{L_{min}^2}; \qquad (22)$$

$$\langle\vec{V}\rangle = 0;\ \langle V_i(t)V_j(t_1)\rangle = 2\nu_E\delta_{ij}\delta(t-t_1);\ \left\langle V_j\frac{\partial u_i}{\partial x_j}\right\rangle = -\nu_E\Delta u_i:\ p = \left(\frac{\eta}{3}+\varsigma\right)div\vec{u}$$

$\eta; \varsigma$ - is the shear and volume viscosity (Chefranovs,2019;2020

Exact regular for all **t>0** explicit solution of Eq. (21) for the case of zero homogeneous friction $\mu = 0;\ \vec{B} = \int_0^t dt_1\vec{V}(t_1);\ P(\vec{B}) = \frac{exp\left(-\frac{\vec{B}^2}{2\langle\vec{B}^2\rangle}\right)}{\left(2\pi\langle\vec{B}^2\rangle\right)^{n/2}};\ \langle\vec{B}^2\rangle = 2nt\nu_E$:

$$\langle u_i(\vec{x};t)\rangle = \int d^nBP(\vec{B})\,u_i(\vec{x}-\vec{B};t) \qquad (23)$$

The predictability problem in any numerical simulation of the Navier-Stokes equations:

For the case of regularization only due to the homogeneous friction the exact solution is obtained by transform in the exact explicit solution for velocity (11): $\boldsymbol{u_i(x;t)} \rightarrow \boldsymbol{u_i(\vec{x};\tau(t));\tau(t)} = \frac{\boldsymbol{1-exp(-\mu t)}}{\boldsymbol{\mu}}$ (24)

This solution is regular for all **t>0** only in the case:

$$\boldsymbol{\mu > \mu_{th} = \frac{1}{t_C}; DNS\ mesh - size\ new\ stability - predictability\ condition}:$$

$$\boldsymbol{L < Lth = \sqrt{t_C \nu}_E \sim Re^{-1/2}} \quad (25)$$

The range of applicability to the exact solutions

1. L. Euler (1757) (see Physica D, **237**, 825 (2008)) consider the case of zero balance of forces in RHS of Euler's equations when $\boldsymbol{\vec{u} = const}$:

$$\boldsymbol{\vec{f} = \vec{\nabla} p}$$

The generalization of Euler's solution in the case (24), but with $\boldsymbol{\vec{u} \neq const}$ for homogeneous Euler's equation:

$$\boldsymbol{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = 0}; \qquad \boldsymbol{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0; i = 1, 2, \ldots n}$$

2. The exact solution to the 3-D Navier-Stokes equations (N-S-E) which is singular at the finite time for the case of exactly zero shear viscosity $\eta = 0$

Solution to N-S-E are conserved there smoothness only over a finite time interval $\mathbf{0 \leq t \leq t_C}$:

$$\widehat{\boldsymbol{det\,A}} \equiv \boldsymbol{det}\left(\boldsymbol{\delta_{ij}} + \boldsymbol{t}\frac{\partial \boldsymbol{u_{0i}}(\vec{\boldsymbol{x}})}{\partial \boldsymbol{x_j}}\right) > \mathbf{0}$$

The exact solution to the 3-D Navier-Stokes equations for the non-equilibrium case

$$\boldsymbol{p} = \left(\frac{\boldsymbol{\eta}}{\mathbf{3}} + \boldsymbol{\varsigma}\right)\boldsymbol{div\vec{u}}$$

$\boldsymbol{\eta} \rightarrow \mathbf{0}$; $\boldsymbol{\varsigma}$ - is the shear and volume viscosity (Chefranovs,2019;2020);

$\boldsymbol{p} = (\boldsymbol{\varsigma})\boldsymbol{div\vec{u}}$ **- Zel'dovich, Raizer, Physics of shock waves and high temperature hydrodynamic phenomena, Moscow 1963**

3. The dispersion-less Kadomtsev-Petviashvili (dKP) equation- universal equation, describing the propagation of non-linear waves in acoustics, plasma physics and hydrodynamics. S. Manakov (2011) consider the limit of small initial amplitudes for solution and obtain that for dimensions n>3 the breaking of wave is absent.

dKP: $$\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}\right) = -\frac{c}{2}\sum_{i=2}^{n} \frac{\partial^2 u_1}{\partial x_i^2}; u_1(t = t_0; \vec{x}) = u_{10}(\vec{x})$$

$$t_B/t_0 = (1 - (n-3)t_C/2t_0)^{2/(n-3)}$$

For $n > n_{th} = 3 + 2t_0/t_C$, as for homogeneous friction $\mu > \mu_{th} = \frac{1}{t_C}$

4. In the case $\vec{u} = U_S \vec{\nabla} f; \mu = -\gamma_0$:

The n-D Sivashinsky equation: $\frac{\partial f}{\partial t} - \frac{U_S}{2}\left(\vec{\nabla} f\right)^2 = \gamma_0 f$

5. The n-D Kadar-Parisi-Zhang equation in the homogeneous case $\boldsymbol{\xi(\vec{x};t)=0}$ in the case of initial potential velocity field as for n-D Sivashinsky Eq :

n − D KPZ: $$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{t}} = \boldsymbol{\nu \Delta h} + \frac{\lambda}{2}\left(\vec{\boldsymbol{\nabla}}\boldsymbol{h}\right)^2 + \boldsymbol{\xi(\vec{x};t)}$$

6. For the case of inertial motion of fluid particles with zero acceleration $\boldsymbol{\vec{a} = D\vec{u}/dt = 0}$
the exact explicit solution (16) of n-D continuity equation gives also an example to the exact solution of the Liouville equation for the common non – Hamiltonian non-equilibrium systems:

$\boldsymbol{n - D\ Liouville's}$: $$\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \sum_{i=1}^{n}\left(\frac{\partial \boldsymbol{\rho u_i}}{\partial \boldsymbol{x_i}} + \frac{\partial \boldsymbol{\rho a_i}}{\partial \boldsymbol{u_i}}\right) = \boldsymbol{0}$$

Turbulence theory
One-point moments:
The problem of Onsager's dissipative anomaly

$$\boldsymbol{\varepsilon_n = lim_{\nu_E \to 0}\, \nu_E \frac{1}{L^n} \int d^n x \left(\left\langle \frac{\partial u_i}{\partial x_j}\right\rangle\right)^2 = f(Re)?\,; n = 1, 2, 3, \ldots}$$

$$\boldsymbol{\varepsilon_1 = lim_{\nu_E \to 0}\, \nu_E \frac{1}{L} \int_{-\infty}^{\infty} dx_1 \left(\frac{\partial \langle u_1(x_1;t)\rangle}{\partial x_1}\right)^2 =?\ \Omega_n^m \equiv \frac{1}{L^n} \int d^n x \left(\frac{\partial \langle u_i\rangle}{\partial x_j}\right)^m \propto Re^{\frac{n}{2}\left(\frac{2m}{3}-1\right)};}$$

Its exact resolution due to the solutions (17), (22): $\boldsymbol{\varepsilon_1 \propto Re^{-\frac{5}{6}}; n = 1}$

$$\boldsymbol{\varepsilon_2 \propto Re^{-\frac{2}{3}}; n = 2; \varepsilon_3 \propto Re^{-\frac{1}{2}}; n = 3; \ldots .\ ; \varepsilon_6 \propto O(1)\,; n = 6}$$

A. A. Migdal(2023):vortex sheet $\boldsymbol{\varepsilon_3 \propto Re^{-\frac{1}{2}}}$; ***Kelvinon -vortex*** $\boldsymbol{\varepsilon_3 \propto O(1)}$;

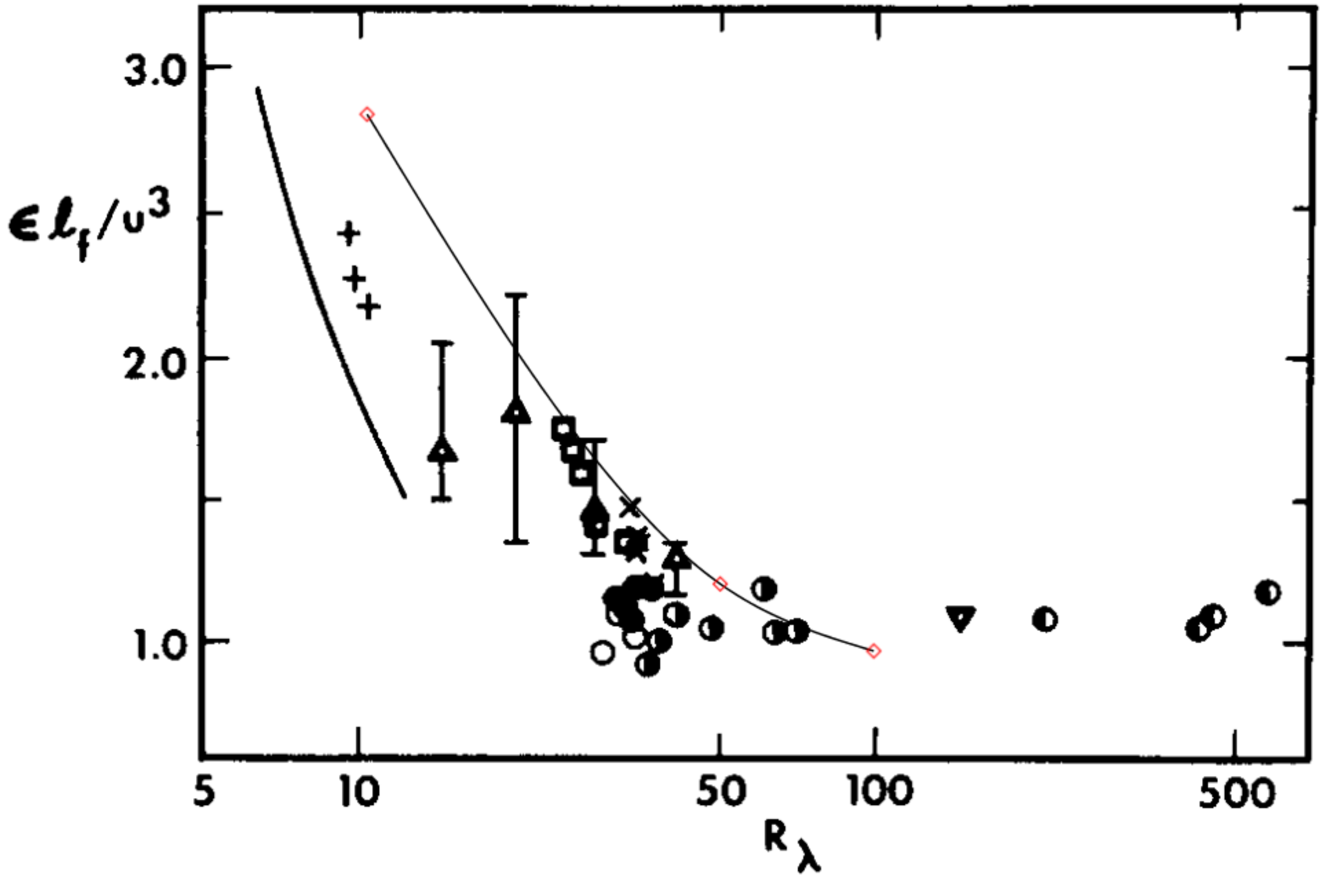


Fig. 1

Data represented in K. R. Sreenivasan (Phys. Fluids 1984) on the dissipation of turbulent energy in grid turbulence for the square-mesh configuration.

Black line in the region of Re (10-100) is obtained solution $\boldsymbol{\varepsilon_3 \propto Re^{-1/2}}$.

Turbulence micro-scales and intermittency

$$\Omega_n^m \equiv \frac{1}{L^n}\int d^n x \left(\frac{\partial\langle u_i\rangle}{\partial x_j}\right)^m \propto Re^{\frac{n}{2}\left(\frac{2m}{3}-1\right)}$$

$$L_{Knew} = L(n=3) \sim Re^{-5/8}, n = 3;$$

$L(n=3) > L(n=6) = L_K \sim Re^{-3/4} >$

$L_d \sim Re^{-1}$**- V. Yakhot, 2005;**

L(n=3)<$L_T \sim Re^{-1/2}$ - Solar plasma 2022

$L(n=3) < L_N \sim Re^{-0.3}, E.A. Novikov\ PRL 1993$

$F_n = \frac{\Omega_n^4}{\left(\Omega_n^2\right)^2}$ **-Flatness;** $F_1 \propto Re^{\frac{1}{2}}; F_2 \propto Re;$

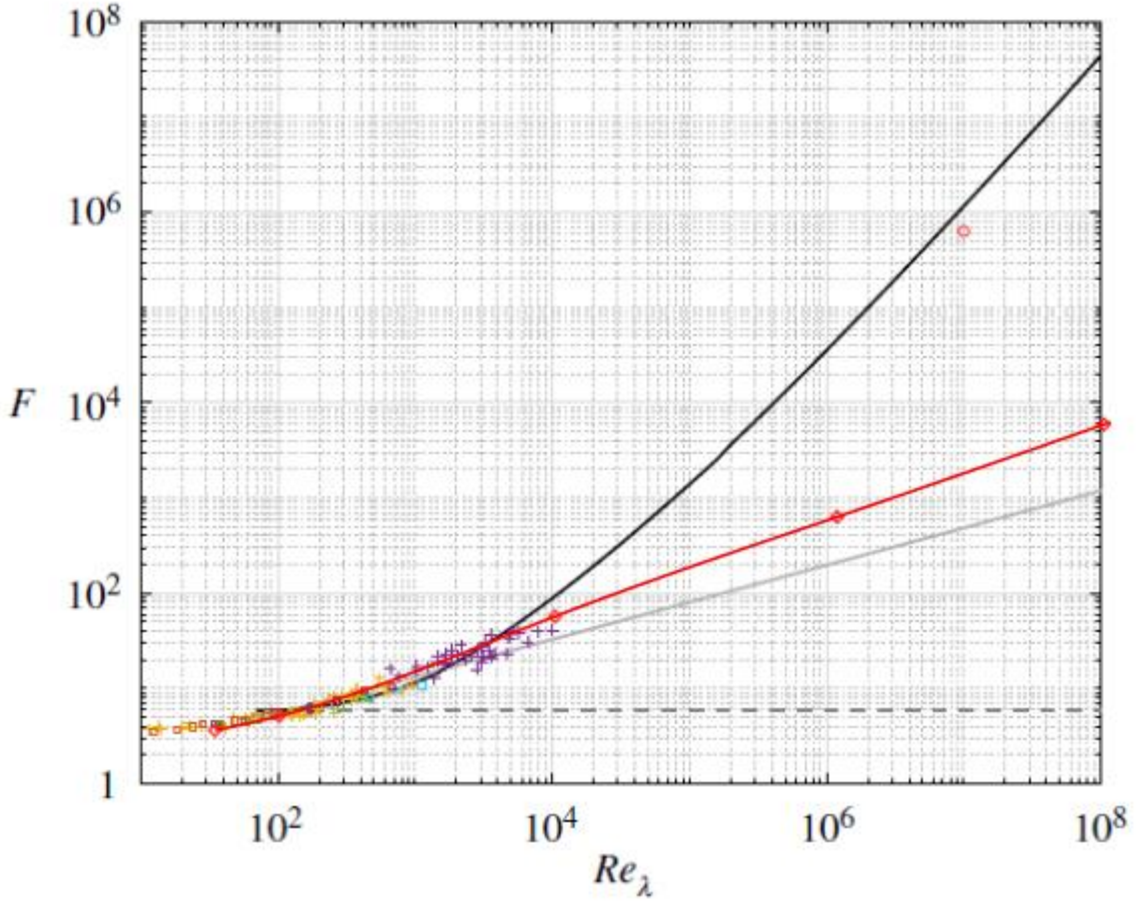


Fig.2

Is taken from G. E. Elsinga, T. Ishihara, J. C. R. Hunt (Proc. R. Soc. A 2020 – Fig.8) for flatness of the longitudinal velocity gradient. DNS- -open squares;

Experimental data: plus- from K.R. Sreenivasan, R.A. Antonia (1997), T. Ishihara et al. (2007); purple plus – for atmospheric turbulence – from J. C. Wyngaard, H. Tennekes (1970).

Grey dashed line – a constant lognormal distribution; Red line – exact solution

$\boldsymbol{F_1 = 0.57\, Re^{1/2}}$ ..

Structure functions of velocity gradients:

$\boldsymbol{D(r) = \frac{1}{L}\int_{-\infty}^{\infty} dx \left[\frac{\partial\langle u(x+r;t)\rangle}{\partial x} - \frac{\partial\langle u(x;t)\rangle}{\partial x}\right]^{m}}$ **; statistics of velocity gradients in problems of mixing, cloud formation, particles clustering, cavitation, flame extinction and etc.** In the limit $\boldsymbol{t \to t_C, \nu_E \to 0; \nu_E \neq 0}$ which replaces the usually considered stationary limit $\boldsymbol{t} \to \infty$:

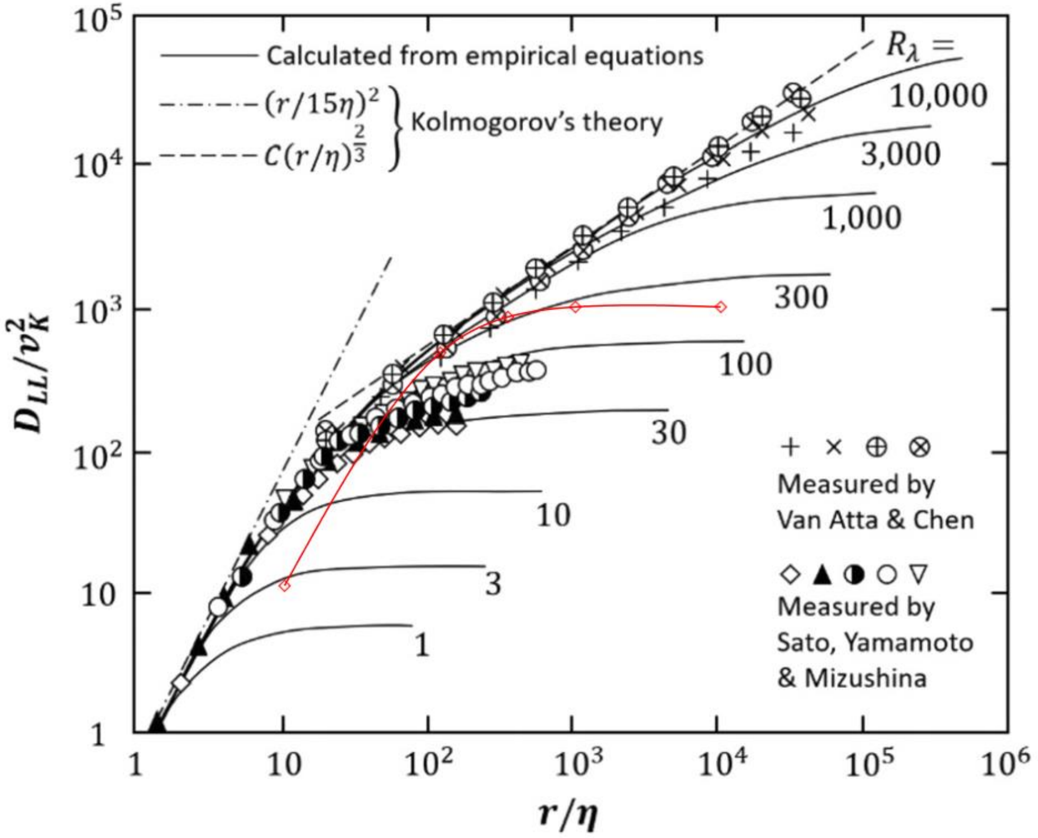


Fig.3
Is taken from the left side of Fig.12 in John Z. Shi paper (Phys. Fluids, **33**, 041301 2021). Red line is obtained for the second order function $D_2(r)$ from exact solution (11) in its regularized form (22).

Structure functions for arbitrary order m:

$$D_2\left(\frac{r}{\sqrt{t_C\nu}}\right)=A_2\left(1-{}_1F_1\left(\frac{1}{6};\frac{1}{2};-\frac{r^2}{8t_C\nu}\right)\right);\ {}_1F_1=F\propto r^{-\frac{2}{3}}exp(-\frac{r^2}{8t_C\nu}),r>>\sqrt{t_C\nu}=L/Re^{1/2}$$

$$A_2=\frac{\Gamma(1/6)A_1^2}{\pi(2t_C\nu)^{1/6}L};A_1=2u_0'(x_M)\sqrt{\pi/3}\Phi(0)\left(\frac{12}{t_C(\gamma+1)u_0'''(x_M)}\right)^{1/3};u_0'\equiv\left(\frac{du_0}{dx}\right)_{x=x_M};$$

$$\Phi(0)=\frac{\sqrt{\pi}}{3^{2/3}\Gamma(2/3)}\approx 0.629;F\,(a;b;z)\equiv 1+\frac{a}{b1!}z+\frac{a(a+1)}{b(b+1)2!}z^2+\ldots$$

F

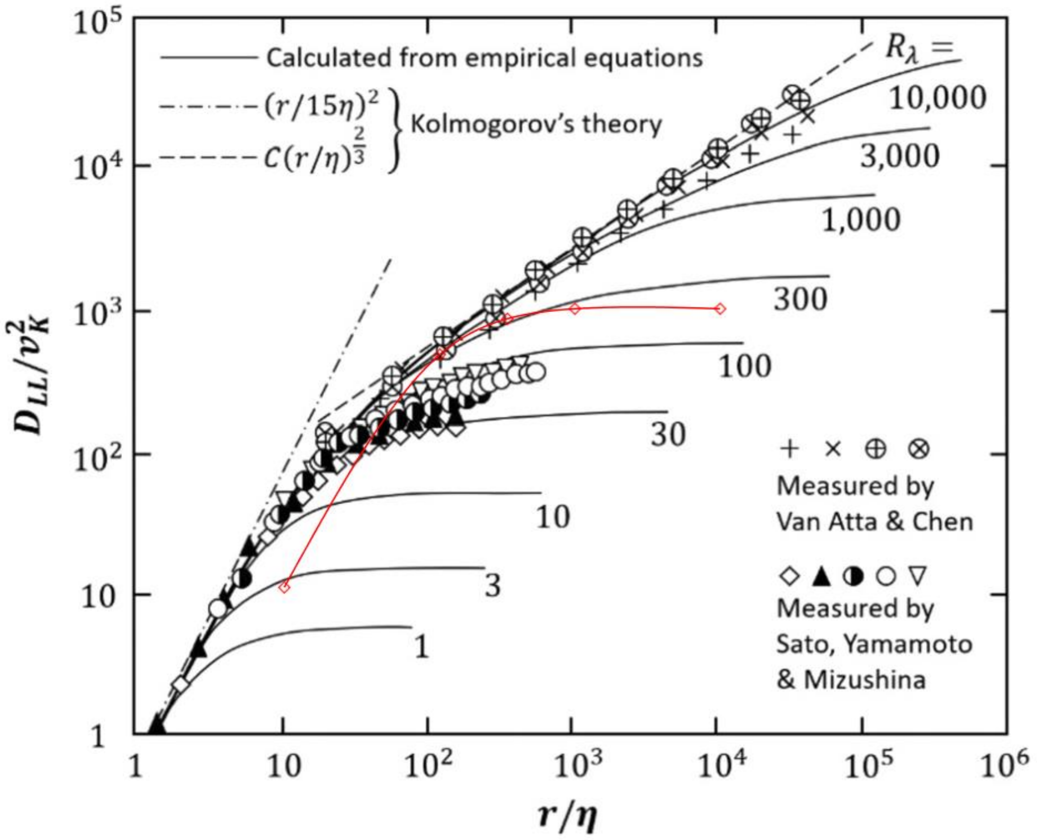
$$D_{2m}=A_1^{2m}(t_C\nu)^{-\frac{4m-3}{6}}\Gamma^{2m}\left(\frac{1}{3}\right)\frac{I_{2m}\left(\frac{r}{\sqrt{t_C\nu}}\right)}{(2\pi)^{2m}}/L;I_{2m}\equiv\int_{-\infty}^{\infty}dx_1\left[F\left(\frac{1}{3};\frac{1}{2};-\frac{\left(x_1+\frac{r}{\sqrt{t_C\nu}}\right)^2}{4}\right)-F\left(\frac{1}{3};\frac{1}{2};-\frac{x_1^2}{4}\right)\right]^{2m}$$

$$D_3(x)=2A_3\int_{x_{1M}}^{\infty}dyF^2\left(\frac{1}{3};\frac{1}{2};-\frac{y^2}{4}\right)\left(F\left(\frac{1}{3};\frac{1}{2};-\frac{(y+x)^2}{4}\right)-F\left(\frac{1}{3};\frac{1}{2};-\frac{(y-x)^2}{4}\right)\right);$$

$$A_3\sim {A_1}^3Re^{1/2};x=\frac{r}{\sqrt{t_C\nu}};x_{1M}=(x_M+t_Cu_0(x_M))/\sqrt{t_C\nu}\ ;D_3\sim -x,x\ll 1;D_3\sim O\left(x^{-1/3}e^{-\frac{x^2}{4}}\right),x\gg 1$$

Millionschikov1941: $D_3\sim xF(5;\frac{7}{2};-\frac{x^2}{4})$

Structure function of velocity



$$D_{Vn}(r) = \frac{1}{L}\int_{-\infty}^{\infty} dx(u(x+r;t) - u(x;t))^n$$

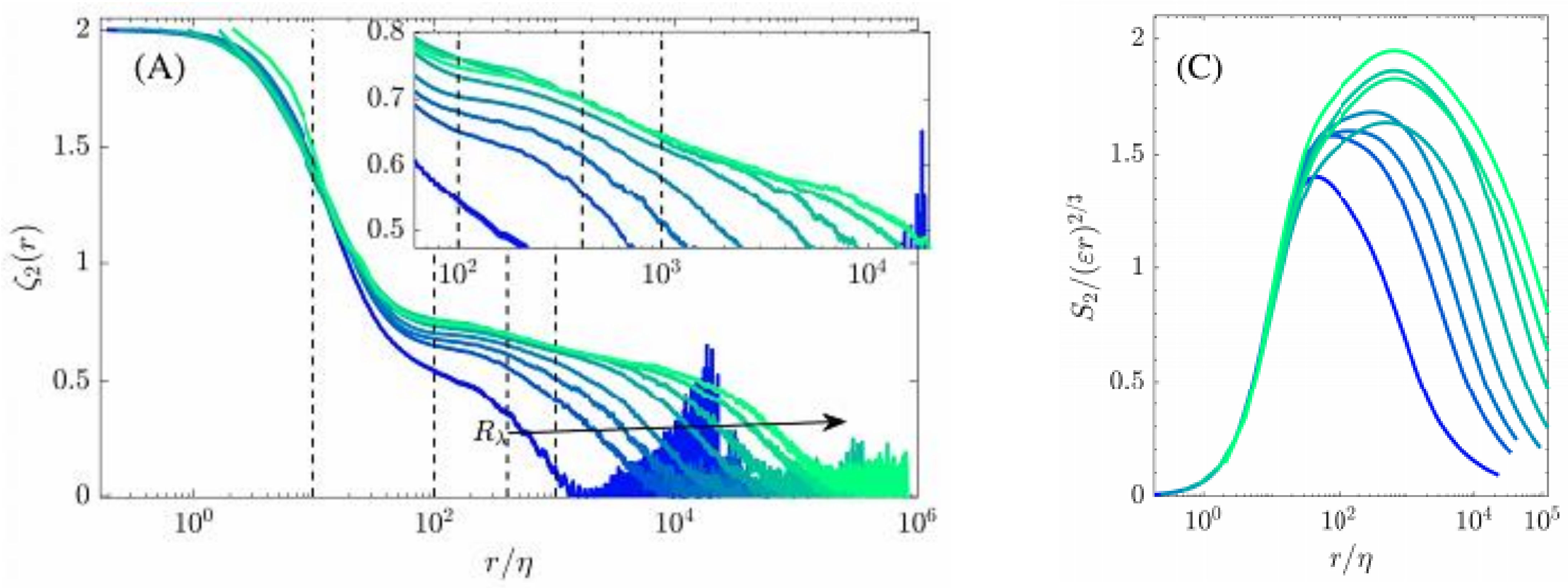
$$D_{V2K}(r) \sim r^{\frac{2}{3}} \rightarrow E_K(k) \sim k^{-\frac{5}{3}};$$

$$D_{2K}(r) \sim r^{-\frac{4}{3}} \leftarrow k^2 E_K \sim k^{\frac{1}{3}}$$

$$D_2(r) \sim 1 - r^{-\frac{2}{3}} \exp\left(-\frac{\boldsymbol{r^2}}{\boldsymbol{8t_C\nu}}\right)$$;-red line

$$\boldsymbol{r >> \sqrt{t_C \nu}}$$

Thus, no single power-law exponent is useful in the case of unsteady turbulence (see also experimental data C. Kuchler, et al.2022)



C. Kuchler, E. Bodenschatz, G. P. Bewley " Scaling in decaying turbulence at high Reynolds", arXiv; 2006.10993v2 [physics.flu-dyn] 4 Mar 2022 (Fig. 1A; Fig. 1C):

Part 2:Explicit form of the Riemann exact solution to the 1-D Euler equations:

1.

Velocity $$\boldsymbol{V(x,t) = \int_{-\infty}^{\infty} d\xi V_0(\xi)(1 + t\frac{du_0(\xi)}{d\xi})\delta(\xi - x + tu_0(\xi))},\qquad (2.1)$$

$$\boldsymbol{u_0(\xi) = V_0(\xi) \pm c_0(\xi)};$$

$$\boldsymbol{c_0 - initial\ velocity\ of\ sound\ in\ the\ politropic\ medium}$$

Density $$\boldsymbol{\rho(x,t) = \int_{-\infty}^{\infty} d\xi \rho_0(\xi)(1 + t\frac{du_0}{d\xi})\delta(\xi - x + tu_0(\xi)); \rho_0(x) = \rho(x, t = 0)}\qquad (2.2)$$

Collapse time: $$\boldsymbol{1 + t(\frac{dV_0(x)}{dx} \pm \frac{dc_0(x)}{dx}) = 0; t_0 = \frac{1}{max\left|\frac{du_0(x)}{dx}\right|}; u_0 \equiv V_0 \pm c_0}\qquad (2.3)$$

For the initial velocity distributions $\boldsymbol{V_0(x) = a\,exp(-x^2/2x_0^2)}$

$\boldsymbol{t_C = \frac{2x_0\sqrt{e}}{a(\gamma+1)}}$ **- time of the solution collapse** **(2.4)**

Energy dissipation in the compressible medium

Integral kinetic energy of the turbulent flow of a compressible medium

$$\boldsymbol{E_C = \frac{1}{2L}\int_{-\infty}^{\infty} dx\,\rho(x;t)V^2(x;t)}$$

$$\boldsymbol{\frac{1}{\rho_\infty}\frac{dE_C}{dt} = -I_D + I_P; I_D = \nu_D\Omega_2 = \frac{\nu_D}{L}\int_{-\infty}^{\infty} dx\left(\frac{\partial V(x;t)}{\partial x}\right)^2; I_P = \frac{1}{\rho_\infty L}\int_{-\infty}^{\infty} dx p(x;t)\frac{\partial V(x;t)}{\partial x}; \nu_D = (\frac{4\eta}{3} + \varsigma)/\rho_\infty}$$

For politropic medium $\boldsymbol{V(x,t) = \int_{-\infty}^{\infty} d\xi V_0(\xi)\left(1 + t\frac{(\gamma+1)}{2}\frac{dV_0}{d\xi}\right)\delta\left(\xi - x + t\left(\pm c_\infty + \frac{(\gamma+1)}{2}V_0(\xi)\right)\right)}$

$$\boldsymbol{p(x;t) = \int_{-\infty}^{\infty} d\xi p_0(\xi)\left(1 + t\frac{(\gamma+1)}{2}\frac{dV_0}{d\xi}\right)\delta\left(\xi - x + t\left(\pm c_\infty + \frac{(\gamma+1)}{2}V_0(\xi)\right)\right)}$$

$$\boldsymbol{c_0(x) = c_\infty \pm \frac{(\gamma-1)}{2}V_0(x); p_0(x) = p_\infty\left(1 \pm \frac{(\gamma-1)}{2c_\infty}V_0(x)\right)^{\frac{2\gamma}{\gamma-1}}}$$

Intermittency, dissipation fluctuations and its universal spectral power-law -2/3

$\boldsymbol{\varepsilon_C = -I_D}$ because for politropic medium $\boldsymbol{I_P = 0}$

Local energy dissipation rate and enstrophy: $\boldsymbol{\varepsilon(x,t) = \frac{\nu}{2}\left(\frac{\partial V(x,t)}{\partial x}\right)^2}$;

$$\boldsymbol{I_D \propto \Omega_2 \propto \frac{1}{L}\int_{-\infty}^{\infty} dx\varepsilon(x;t),}$$

Structural function: $\boldsymbol{S_D(r) = -\langle \varepsilon(x+r;t)\varepsilon(x;t)\rangle + \langle \varepsilon\rangle^2}$

$\boldsymbol{(r; t = t_C) \propto r^{-1/3}}$ in the limit $\boldsymbol{r \to 0; t \to t_C}$ (2.5)

Spectrum $\boldsymbol{E_D(k) = C_D k^{-2/3}\, exp\left(-t_C k^2 \nu\right)}$

In the inertial range of scales $\boldsymbol{L^{-1} << k << l_\nu = (2t_C\nu)^{-1/2}}$

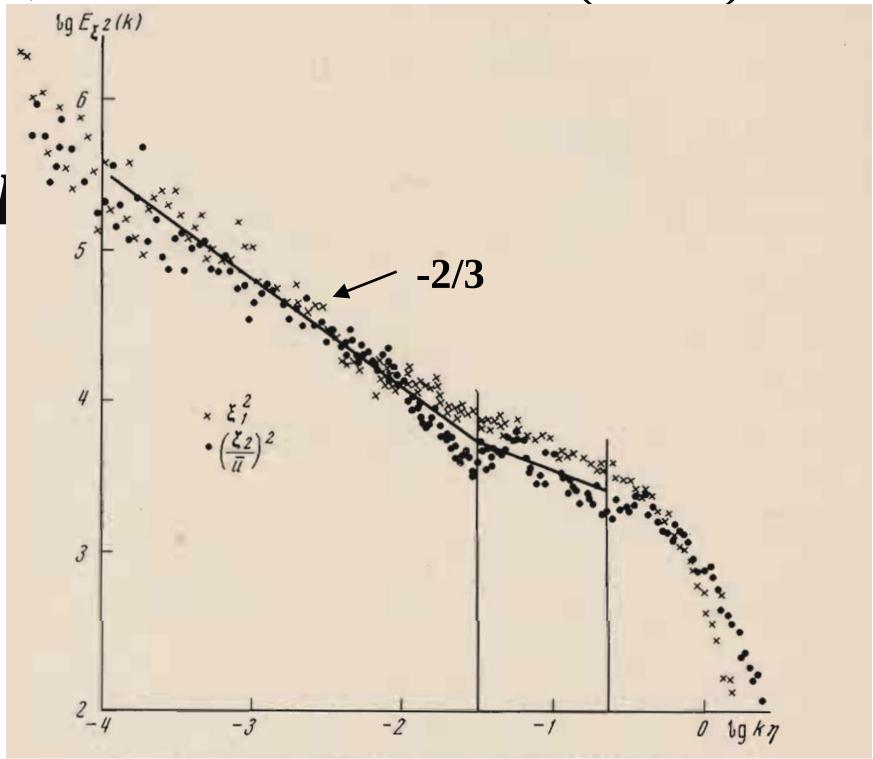
(2.6) .

Intermittence and dissipation fluctuations

Up to now days, for the scaling law -5/3, corrections are introduced related to the heuristic description: $\boldsymbol{E(k) \propto \langle \varepsilon \rangle^{2/3} k^{-\frac{5}{3}} (Lk)^{-q}}$, where $\boldsymbol{q = \frac{\mu}{9}}$ and $\boldsymbol{L}$ is the integral turbulence scale.

$\boldsymbol{\mu = \frac{\log \alpha}{\log \beta}; \frac{\lambda_1}{l_1} = \ldots = \frac{\lambda_j}{l_j} = \alpha << 1; \frac{\lambda_2}{\lambda_1} = \ldots = \frac{\lambda_j}{\lambda_{j-1}} = \beta << \alpha; \; 0 < \mu < 1}$, Novikov-Stewart (1964).

$\boldsymbol{E_D(k) \propto k^{-1+\mu}}$, **our exact solution:** $\boldsymbol{E_D(k) \sim k^{-\frac{2}{3}}}$, I



$\boldsymbol{S_D(r) = \langle (\varepsilon(\vec{x} + \vec{r}; t) - \varepsilon(\vec{x}; t))^2 \rangle \propto r^{-\mu}}$

$\boldsymbol{\mu \approx 0.38 \pm 0.05}$ ond, Stewart (1965);$\boldsymbol{k = 0.01cm^{-1} - 10cm^{-1}}$

$\boldsymbol{\mu \approx 0.33}$ Kholmyansky (1972)

$\boldsymbol{\mu = 1/3}$ Exact solution (Chefranovs Phys. Fluids, 2021)

Spectra of the squared derivative of the wind speed

Universal turbulence energy spectrum law -8/3

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} drR(r)\, exp(-ikr) = \frac{1}{2\pi L} I(k)I^*(k)\ ;\ e = \int_0^{\infty} dkE(k)$$

$$I(k) = \int_{-\infty}^{\infty} dxV_0(x)\frac{\partial S}{\partial x} exp(ikS(x,t)) = \frac{i}{k}\int_{-\infty}^{\infty} dx\frac{dV_0}{dx} exp(ikS)\,; I^*(k) \equiv I(-k)$$

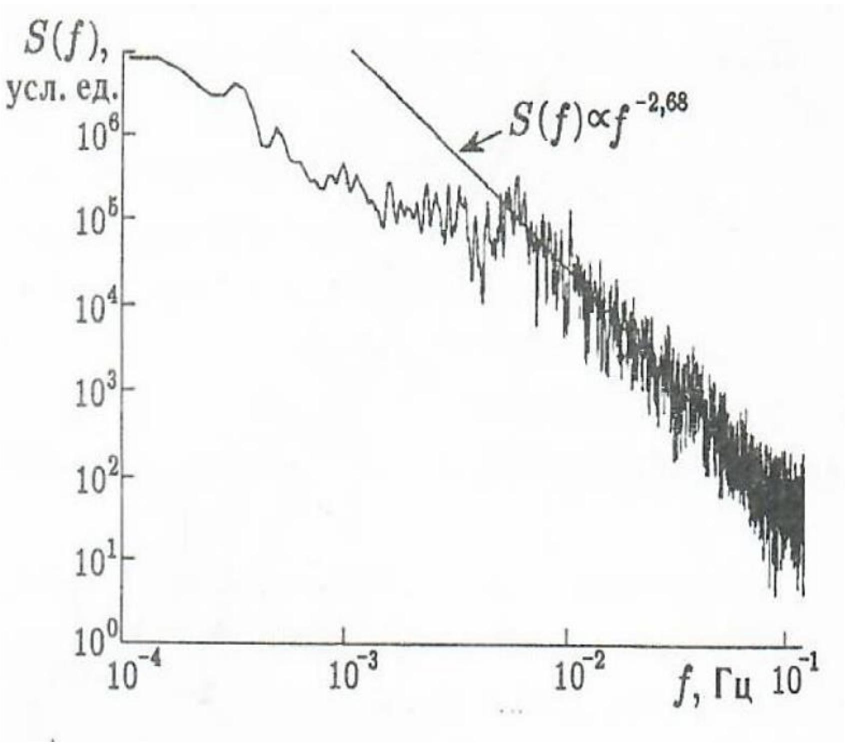
$$S(x;t) = x + \frac{(\gamma+1)t}{2}V_0(x); A(x,t) \equiv \frac{\partial S}{\partial x}$$

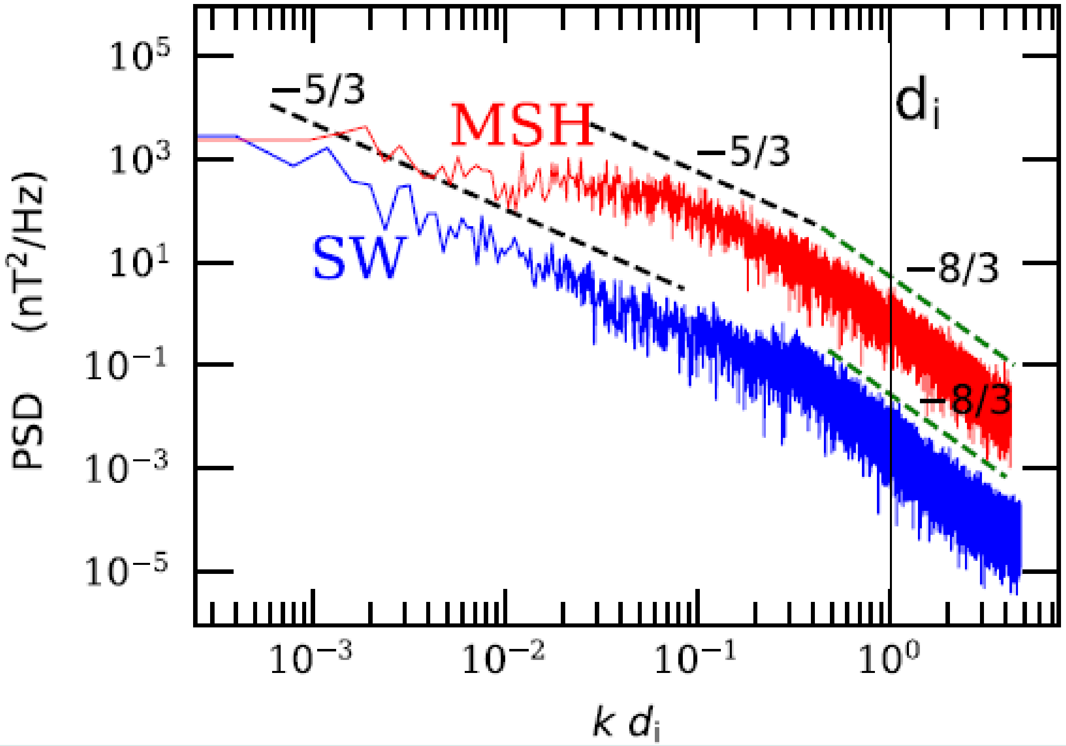
In the limit $kL >> 1; Re \gg 1$ *in the inertial subrange of scales* and near the collapse $t \to t_C$

$$E(k) = C_E k^{-8/3} \qquad (2.7)$$

$$C_E = \frac{2^{5/3}}{L}\left(\frac{dV_0}{dx}\right)_{x=x_M}^{8/3}\left(\frac{d^3V_0}{dx^3}\right)_{x=x_M}^{-2/3}\Phi^2(0)$$

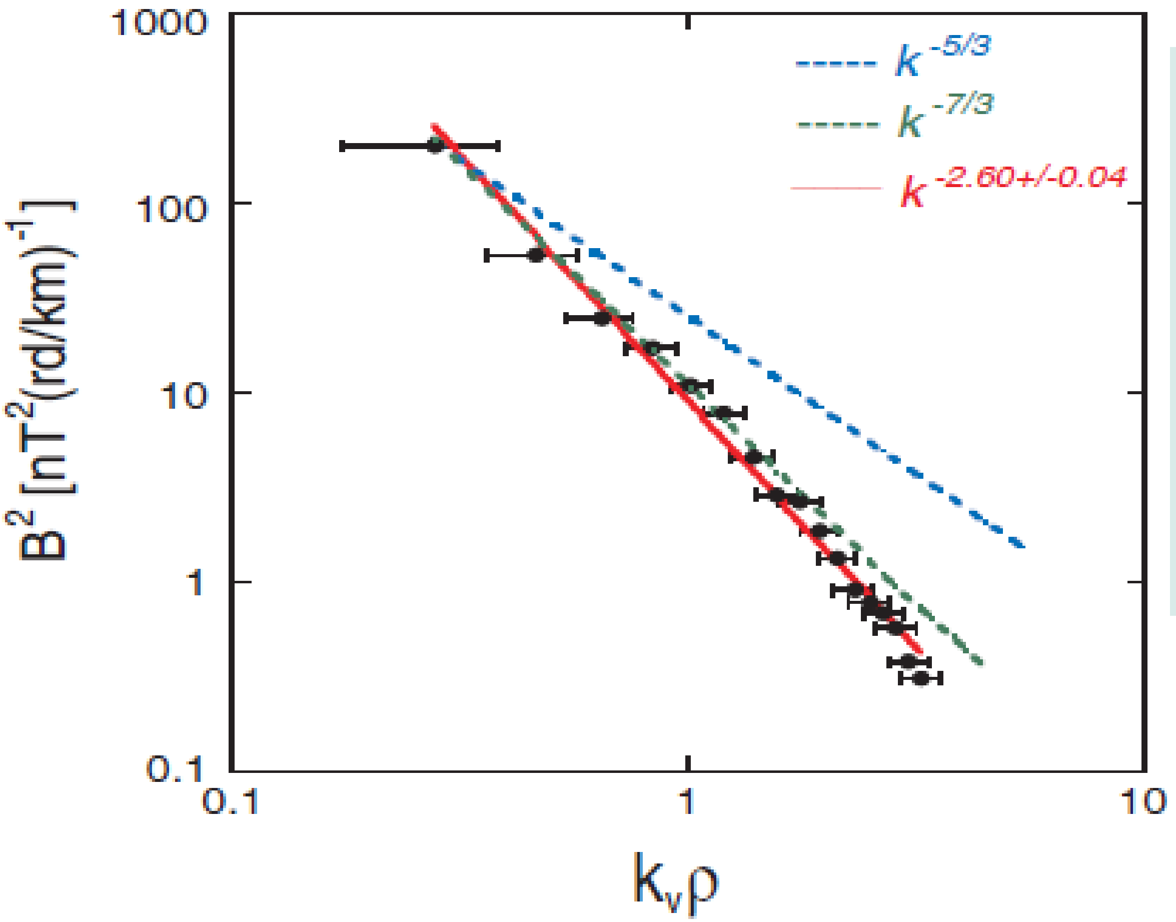
$\Phi(0) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} d\,u\,cos(\frac{u^3}{3}); \Phi(z) = \sqrt{\pi}Ai(z)$ - the Airy function $\Phi(0) = \frac{\sqrt{\pi}}{3^{2/3}\Gamma(2/3)} \approx 0.629.$



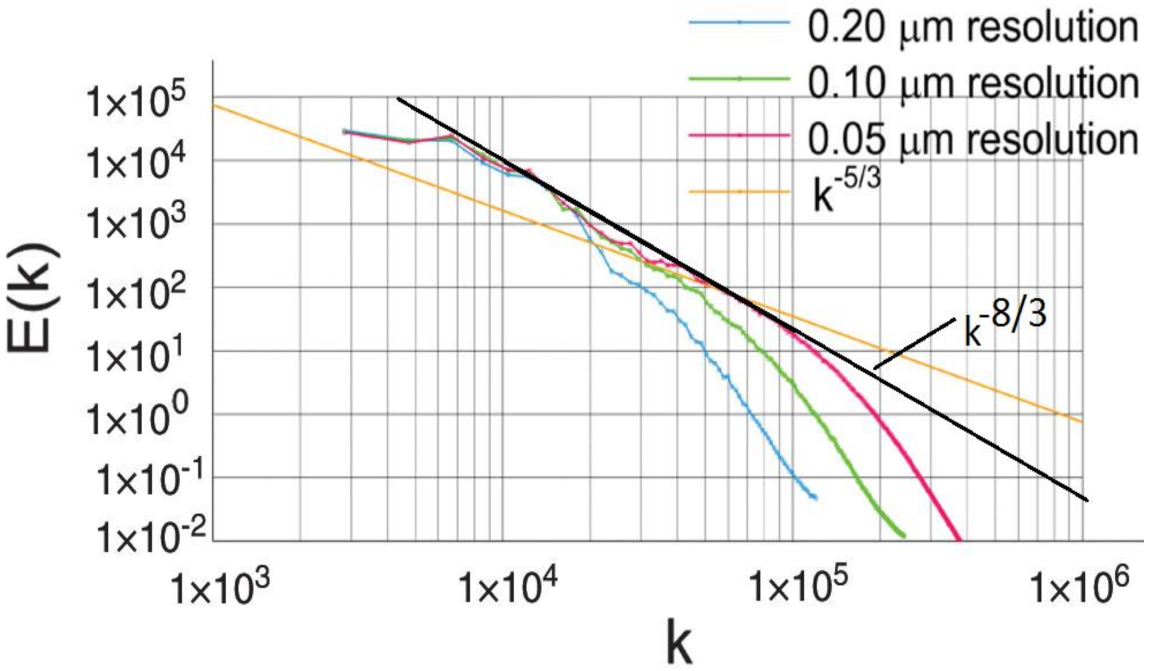


Electric field pulsation spectrum of aero-electric structures
in the surface layer of atmosphere.
Spectrum of the electric-field pulsations.

Magnetic field turbulence spectra for the solar-wind (SW) in blue
and magnetosheath (MSH) in red..

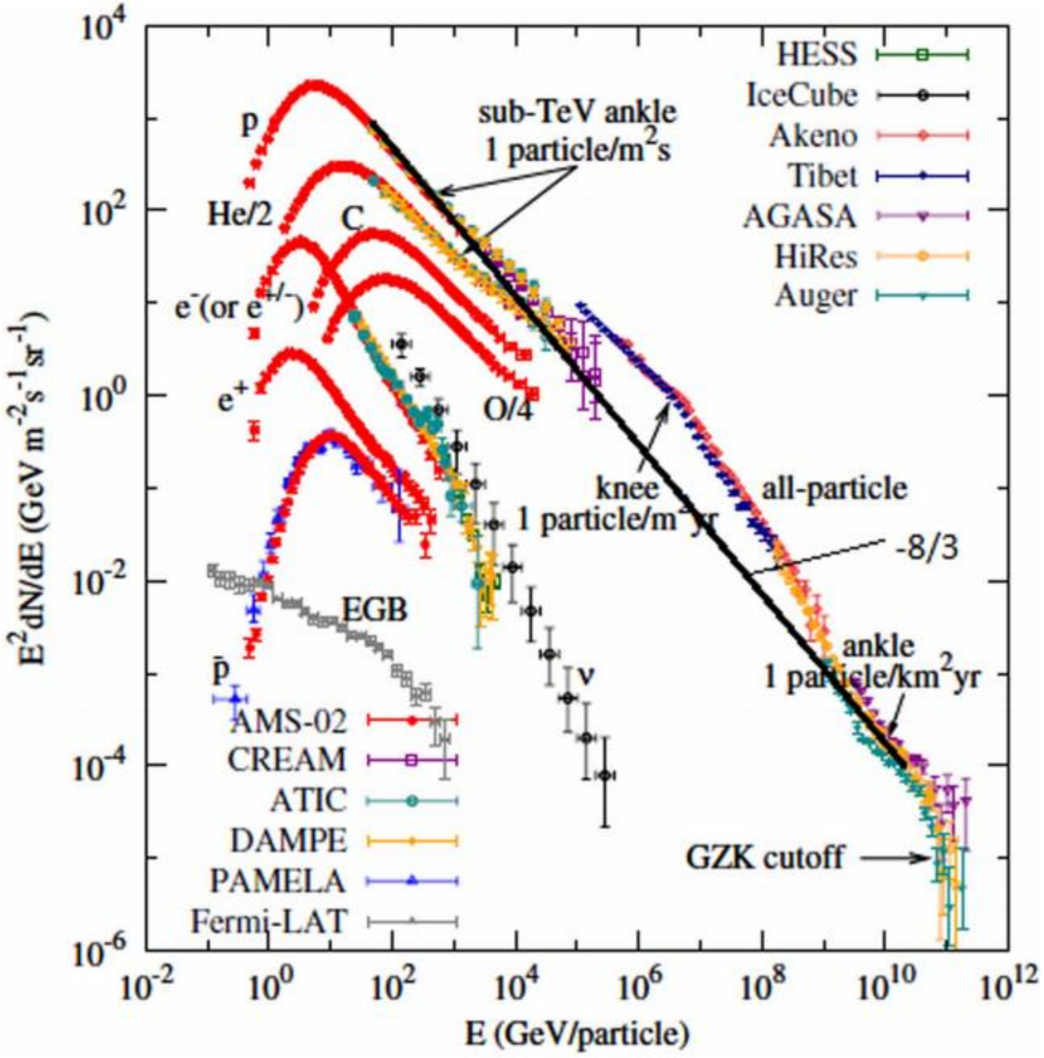


Magnetic field turbulence spectrum for the magnetosheath (when proton Larmor radius $\boldsymbol{\rho} = \mathbf{75}$ km). The red line is a direct fit revealing a power law $\boldsymbol{k}^{-\mathbf{2.6}}$.

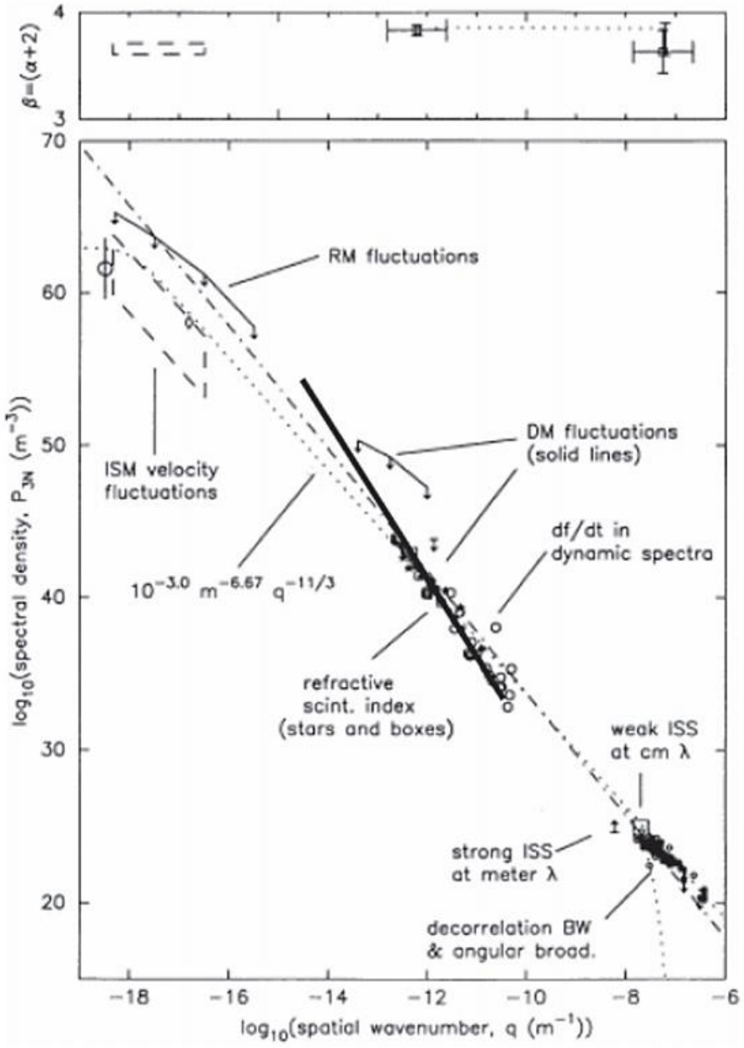


Turbulence kinetic energy spectra on the stagnation stage of fusion implosion at t=1.71 ns

The possibility of arise the helical turbulence with the chiral symmetry breaking on the stages before stagnation is stated on this base. A possible reason for this is a violation of spherical symmetry during the fusion implosion due to the occurrence of rotation of the medium behind the front of a converging spherical shock wave by the instability.



The black solid line represents the energy power-law −8/3



The turbulent spectrum of electron density fluctuations..

Conclusions-Turbulence

The Onsager dissipative anomaly is resolved in the explicit exact analytical form for an arbitrary dimensions. Only in the case of 6-D solution to the homogeneous Euler equations the dependence of the average turbulence energy dissipation rate on the Reynolds number is absent at all. In the 3-D case that dependence is relevant to the observational data in the solar wind turbulence from the Parker Solar Probe (M. E. Cuesta et al. 2022).

Stated that no single power-law exponent is useful in the case of unsteady turbulence for the velocity gradient structure function for the flatness $F \sim Re^{1/2}$

In the inertial interval of the scales in the steady limit an exponent of the power-law -2/3 spectrum of fluctuations in the rate of energy dissipation was obtained, consistent with the observational data.

Conclusions-Hydrodynamics

The obtained of universal turbulence spectrum power-law -8/3 corresponds to the parameters of the turbulent spectra observed in the Earth's and Saturn's magnetospheres, as well as for the Solar Wind, Cosmic Rays and for the turbulence arising during the fusion implosion.

The singularity of the solution in the case of nonlinear wave collapse can be eliminated by taking into account of an arbitrarily small, but non-zero effective viscosity or by the super threshold homogeneous friction. The effect of additional dimensionality transverse to the direction of propagation of a nonlinear wave is similar to the threshold effect of homogeneous friction.

The non-zero viscosity regularization of the unsteady solution to the Helmholtz 3-D vortex equations for an unlimited time gives unexpected positive resolution for the generalization of the Clay problem (www.claymath.org) to the 3-D fluid dynamics in the compressible case.